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THREE-DIMENSIONAL DIGITAL PLANES

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ABSTRACT

Definitions of 3D digital surface and plane are introduced. Many geometric properties of these objects are examined. In particular, it is shown that digital convexity is neither a necessary nor a sufficient condition for a digital surface element to be a convex digital plane element, but it is both necessary and sufficient for a digital surface to be a digital plane. Also algorithms are presented to determine whether or not a finite set of digital points is a (convex) digital plane element.

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Definitions of 3D digital surface and plane are introduced. Many geometric properties of these objects are examined. In particular, it is shown that digital convexity is neither a necessary nor a sufficient condition for a digital surface element to be a convex digital plane element, but it is both necessary and sufficient for a digital surface to be a digital plane. Also, algorithms are presented to determine whether or not a finite set of digital points in a (convex) digital plane element.

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1. Introduction

Development of a theory of three-dimensional (or simply, 3D) digital geometry is essential to research in 3D image processing, which has seen growing interest recently [1-5, 12, 15-18]. Some work has been done already on the subject [8, 10, 11]. This paper is concerned with 3D digital surfaces and planarity.

There are many problems with which 3D digital image processing is concerned. Three problems among them which this paper deals with are the following. The first is when a set of 3D digital points may be considered as a digital surface rather than a digital solid and in particular, as a digital plane. Next is the problem of digitization, that is, methods of representing a 3D continuous surface by a finite set of 3D digital points. The last problem is how to tell whether or not a digital surface element is a representation of a convex plane element.

We define what 3D digital surfaces, planes and plane elements are. Also a method of digitizing 3D continuous surfaces is given. It is shown that the digitization of a plane always results in a digital surface, which is a motivation for our definition of a 3D digital surface.

Geometric properties of digital plane elements are examined. As shown later, digital convexity is neither a necessary nor a sufficient condition for a digital surface element to be a convex digital plane element. That is, a convex digital plane element need not be digitally convex and a digital surface element which is digitally convex need not be a convex digital plane element. Thus, an important property of convex plane elements in 3D Euclidean geometry is not preserved in 3D digital geometry. Next shown is that a digital surface is a digital plane if and only if it is digitally convex. This is a geometric property that carries over from 3D Euclidean geometry.

Next we study a necessary and sufficient condition for a digital surface element to be a digital plane element. This enables us to design algorithms to determine whether or not a set of 3D digital points is a digital plane element and in particular, a convex digital plane element.

In the next section, notations, terminology and definitions are introduced. Also a few known results are stated that are needed in this paper. Section 3 discusses the digital images of planes. In Section 4, geometric properties of digital plane elements are discussed, while those of digital planes are discussed in Section 5. In Section 6, algorithms are presented to determine whether or not a set of 3D digital points is a digital plane element and in particular, a convex digital plane element. Also the complexities of these algorithms are analyzed.

(2D) digital line segments

A (2D) digital arc R is a (2D) digital line segment if there is a line segment f whose digital image is R, that is, R=I(f) (R=I'(f)).

Below we state as lemmas a few known results that are needed in this paper.

Lemma A (Theorem 5 in [9])

The following statements are equivalent:

(i) A digital region is digitally convex.

(ii) A digital region has the chord property.

(iii) The convex hull of a digital region contains no digital point which is not a point of the digital region.

Lemma B (Theorem 6 in [7])

A digital region is digitally convex if and only if for any two of its points, there is a digital line segment in the digital region that connects the two.

Lemma C (Theorem 7 in [8])

A digital solid is digitally convex if and only if it has the chordal triangle property.

Lemma D (Theorem 4 in [10])

A digital arc is a digital line segment if and only if there are two coordinate planes such that the projections of the digital arc on them are 2D digital arcs and have the chord property.

Now we introduce definitions of digital surface and digital plane.

Digital surfaces

A 26-connected subset S of D is a <u>digital surface</u> if for each point $d = (i,j,k)$ of S, the following holds:

In at least two of $S \cap D_{x=i}$, $S \cap D_{y=j}$ and $S \cap D_{z=k}$, point d has at most two 8-neighbors. When it has two, they are not mutually 8-neighbors. If in one of them, say $S \cap D_{z=k}$, point d has more than two 8-neighbors or two 8-neighbors that are mutually 8-neighbors, then both $d'=(i,j,k-1)$ and $d''=(i,j,k+1)$ are not points of S.

A point $d=(i,j,k)$ of a digital surface S is a <u>boundary point</u> of S if it has only one 26-neighbor in $S \cap D_{x=i}$, $S \cap D_{y=j}$ or $S \cap D_{z=k}$. A point d of S is an <u>interior point</u> of S if it is not a boundary point.

A simple <u>digital surface</u> is an infinite set of digital points which is a digital surface with no boundary points. A <u>closed simple digital surface</u> is a finite digital surface with no boundary points. A <u>digital surface element</u> is a finite digital surface whose boundary points are 26-connected.

When a (Euclidean) surface f intersects a coordinate line, there may be one or two digital points that are nearest to the point of intersection. In case there are two, the one inside f is chosen as the nearest, that is, the one that lies on the side opposite the normal vector of the surface.

Digital images of surface elements and simple surfaces

Let f be a surface element or a simple surface. Whenever f intersects a coordinate line, the nearest digital point to the intersection becomes a point of the digital image of f, which is denoted by I(f).

Digital plane elements

A 26-connected digital surface element R is a digital plane element if there is a connected finite subset g of a plane whose digital image is R, that is, $R=I(g)$.

Convex digital plane elements

A digital plane element R is convex if its projections onto the coordinate planes are convex digital regions.

Digital planes

A simple digital surface S is a digital plane if for any given three points of S, there is a subset of S that contains the three points and is a digital convex plane element.

It is easy to see that for any digital surface S, there is a surface f such that $S=I(f)$. It is obvious that the digital image of a surface element or a simple surface is not necessarily a digital surface. However, we will show that the digital image of a plane is a digital surface. We also note that we did not define digital plane as the digital image of some plane. Consider the digital surface $S=\{(i,j,0)|i\varepsilon N\cup N^-, j\varepsilon N^-\}\cup \{(i,j,1)|i\varepsilon N\cup N^-, j\varepsilon N\}$, where N^- is the set of all negative integers and N is the set of all nonnegative integers. There is no plane f whose digital image is S. But for any 26-connected finite subset R of S, there is connected finite subset g of a plane whose digital image is R. Hense, S is a digital plane by our definition.

3. Digital images of planes

In this section we show that the digital image of a plane is a digital surface. Also we state two simple corollaries to this theorem.

Let p be a plane given by equation $ax+by+cz+e=0$, and assume without loss of generality that $0 \le a \le b \le c$ and $0<c$.

Lemma 1

The digital image of plane p may be determined by its crossings of vertical grid lines only (coordinate lines parallel to the Z-axis). Hence, on each coordinate line parallel to the Z-axis, there is exactly one point of the digital image of plane p.

Proof

Let p cross a grid line parallel to the Y-axis, $x=\ell$ and $z=n$, at y', and let $m=\lfloor y' \rfloor$. If $y'=m$, then (ℓ,m,n) is a point of $I(p)$ obtained from its crossing of the grid line $x=\ell$ and $z=n$. It is also a point of p crossing vertical grid line $x=\ell$ and $y=m$. Let $y' \neq m$ and consider the line f given by $by+cz= -(a\ell+e)$, which is the intersection of plane p and the $(x=\ell)$ -plane. Suppose $y'-m < 1/2$. Then (ℓ,m,n) is a point of $I(p)$ determined by its crossing of the grid line $x=\ell$ and $z=n$. Let z' be the intersection of f and vertical grid line $x=\ell$ and $y=m$. Since $b \le c$, $n-z' < 1/2$ and thus point (ℓ,m,n) of $I(p)$ is also obtained from its crossing of a vertical grid line. Next suppose $y'-m \ge 1/2$. Then $(\ell,m+1,n)$ is a point of $I(p)$ obtained from its crossing of the grid line $x=\ell$ and $z=n$. Let z' be the intersection of f and vertical grid line $x=\ell$ and $y=m+1$. Then

$z'-n < 1/2$ and thus point $(\ell,m+1,n)$ of $I(p)$ is also obtained from its crossing of the grid line $x=\ell$ and $z=n$.

Similarily, those points of $I(p)$ that are determined by p crossing coordinate lines parallel to the X-axis are also obtained by its crossing of vertical grid lines.

Since p crosses each vertical grid line exactly once, the digital image of p has exactly one point on each coordinate line parallel to the Z-axis. □

<u>Theorem 2</u>

The digital image of a plane is a digital surface.

<u>Proof</u>

Let p be a plane and suppose that it is given by the equation $ax+by+cz+e=0$. We may assume without loss of generality that $0 \le a \le b \le c$ and $0<c$. Then by *Lemma* 1, $I(p)$ has exactly one point on each vertical grid line and is obtainable from the intersection of p and vertical grid lines.

Let $d=(i,j,k)$ be a point of $I(p)$. Then d is the only point of $I(p)$ on the vertical grid line $x=i$ and $y=j$.

First consider $I(p) \cap D_{x=i}$, and let $d'=(i,j-1,k')$ and $d''=(i,j+1,k'')$ be the only two points of $I(p)$ on vertical lines $x=i$, $y=j-i$ and $x=i$, $y=j+1$, respectively. Since $b \le c$, $0 \le |k-k'|, |k-k''| \le 1$. Thus, d' and d" are the only two 8-neighbors of d and are not mutually 8-neighbors.

Similarly, d has only two 8-neighbors in $I(p) \cap D_{y=j}$ and they are not mutually 8-neighbors.

In $I(p) \cap D_{z=k}$, d may have more than two 8-neighbors but both $(i,j,k-1)$ and $(i,j,k+1)$ are not points of $I(p)$ because

$d=(i,j,k)$ is the only point of $I(p)$ on the vertical grid line $x=i$ and $y=j$. Thus $I(p)$ is a digital surface. □

Corollary 3

The digital image of a convex subset of a plane is a digital surface element.

Corollary 4

If it is connected, the digital image of a connected 'bset of a plane is a digital surface element.

As mentioned before, the digital image of a surface : ɔot necessarily a digital surface. However, it is very likel that the digital image of a surface whose curvature at every point is small is a digital surface. We note that a plane is a surface with zero curvature everywhere and its digital image is a digital surface. But it is also the case that however small the curvature of a surface is everywhere, its digital image may not be a digital surface. As an example, consider the surface q whose intersection with coordinate plane $x=i$, i being any integer, is as shown in Figure 1.

Surface q is obtained by slightly perturbing plane given by $y+z - 1/2 = 0$ so that q intersects coordinate planes $y=-1$ and $y=\varepsilon$ at $z=-1/2+\varepsilon$ and $z=1/2+\varepsilon$, respectively, where ε is a small positive number. The digital image of q is not a digital surface but the curvature of the plane at every point may be as close to 0 as desired.

4. Digital plane elements and their geometric properties

We examine several geometric properties of convex digital plane elements. Many of the results shown are negative in that geometric properties enjoyed by 3D Enclidean convex plane elements are not enjoyed by 3D convex digital plane elements. The most interesting result is that digital convexity is neither a necessary nor a sufficient condition for a digital surface element to be a convex digital plane element. Because of Lemma C, it suffices to show that the chordal triangle property is neither a necessary nor a sufficient condition, which is shown in the following theorem.

Theorem 5

The chordal triangle property is neither a necessary nor a sufficient condition for a digital plane element to be a convex digital plane element, and the same is true for the chord property.

Proof

The finite set of digital points R shown in Fig. 2 is a convex digital plane element. For, obviously the projection of R on each coordinate plane is digitally convex, and it is easy to see that R is the digital image of a finite subset of the plane given by the equation $2y+5z-12.5=0$. But R does not have the chord property and hence it does not have the chordal triangle property either. To see this, consider the chord between $(0,0,2)$ and $(2,5,0)$. The point $(1, 2\,1/2, 1)$ on the chord is not near R.

It is not difficult to see that the finite set of digital points R' in Fig. 3 has the chordal triangle property, hence the chord property also. But it is not only not a convex digital plane element but not even a digital plane element. We note from the proof of Lemma 1 that the projection onto one of the coordinate planes of the digital image of a plane element is one-to-one. Since the projection of R' onto any coordinate plane is not one-to-one, R' is not a plane element. □

Theorem 6

The volume property is a necessary but not a sufficient condition for a digital plane element to be a convex digital plane element.

Proof

Consider the set R' of digital points in Fig. 3, which is not even a digital plane element. Since a finite set of digital points has the volume property if it has the chordal triangle property and R' has the chordal triangle property, R' has the volume property. Hence, the volume property is not a sufficient condition.

Now suppose that a digital surface element R is a convex digital plane element. Then there is a connected finite subset g of a plane p whose sigital image I(g) is R. Let p be given by $ax+by+cz+e=0$ and assume without loss of generality that $0\le a\le b\le c$ and $0<c$. Consider the projection R_z of R onto the (z=0)-plane. Let $H(R_z)$ be the convex hull of R_z. Then by

the definitions of 2-D digital convexity and digital plane element, the set of all digital points of $H(R_z)$ is R_z. Let h be the subset of p such that its projection h_z onto the (z=0)-plane is $H(R_z)$. Also let h' and h" be the plane elements obtained by translation of h parallel to the Z-axis by 1/2 and -1/2, respectively. Then every point of R lies between h' and h", possibly on h" but not on h'. Let V be the volume bounded by h', h" and the vertical surfaces connecting the boundaries of h' and h". Then every point of R is in V and every digital point of V except those on h' is a point of R. Since V is convex and the convex hull H(R) of R is a subset of V and has no point of h', H(R) contains no digital point not in R. Thus, R has the volume property. □

Theorem 7

It is a necessary but not a sufficient condition for a digital plane element R to be a convex digital plane element that for any two points d_1,d_2 of R, there is a digital line segment A in R whose endpoints are d_1 and d_2.

Proof

Consider the digital surface element R shown in Fig. 2, which is not a digital plane element. It is easy to show that for any two points d_1,d_2 of R, there is a digital line segment whose endpoints are d_1 and d_2.

Now let R be a digital plane element and d_1,d_2 be any two points of R. Consider R_z and R_y which are the projections of R onto the (z=0)-plane and (y=0)-plane, respectively. By definition they are convex digital regions. Let $d_{1,z}$ and $d_{2,z}$ be the points of R_z that are projections of d_1 and d_2, respectively, onto the (z=0)-plane. Since R_z is a convex digital region, there is a digital line segment whose endpoints are $d_{1,z}$ and $d_{2,z}$ by Lemma B. Let f_z be a line on the (z=0)-plane such that the digital line segment in R_z connecting $d_{1,z}$ and $d_{2,z}$ is a subset of the digital image of f_z, $I'(f_z)$. Let f be the plane which is perpendicular to the (z=0)-plane and intersects it at f_z. Similarly, let g be a plane perpendicular to the (y=0)-plane such that a digital line segment in R_y connecting $d_{1,y}$ and $d_{2,y}$ is a subset of the digital image of g_z, the line of intersection of the two planes. Let h be the line of intersection of the two planes f and g. Then d_1 and d_2 are points of the digital image of h, I(h) and the segment of I(h) between d_1 and d_2 is a subset of R. □

<u>Theorem 8</u>

The digital image of a convex plane element is not necessarily a convex digital plane element.

<u>Proof</u>

Consider the triangle T whose vertices are (0,0,0), (2, 1/2, - 1) and (0,2, -1) shown in Fig. 4. Its digital image I(T) is the set of five digital points indicated in the figure. Since its projection onto the (z=0)-plane is not digitally convex, it is not a convex digital plane element. □

5. Digital planes and their geometric properties

Geometric properties of digital planes are studied in this section. The main result is that digital convexity is a necessary and sufficient condition for a simple digital surface to be a digital plane. We first define the distance between a finite set of digital points and a (Euclidean) plane. Let T be a finite set of digital points and p a plane. Suppose $d = (i,j,k)$ is a point of T. The z-distance between d and p, $\text{dist}_z(d,p)$, is defined to be the vertical distance from d to p, that is, if $w = (i,j,z')$ is a point of p, then $\text{dist}_z(d,p) = |z'-k|$; $\text{dist}_x(d,p)$ and $\text{dist}_y(d,p)$ are similarly defined. The distance between T and p is defined by $\text{dist}(d,p) = \min\{\max_{d\varepsilon T}$ $\text{dist}_x(d,p)\}, \max_{d\varepsilon T}\{\text{dist}_y(d,p)\}, \max_{d\varepsilon T}\{\text{dist}_z(d,p)\}\}$.

Lemma 9

Given any digital plane, there is a coordinate plane such that the projection of the digital plane onto it is one-to-one and onto the set of its digital points.

Proof

Let the simple digital surface S be a digital plane. Suppose that the projections of S onto each coordinate plane are bounded by two parallel lines. Then S is bounded by a parallelepiped and so is finite, which is a contradiction. Therefore, there is a coordinate plane, say the (z=0)-plane, such that S_z, the projection of S onto it, is not bounded by any two parallel lines.

Choose d_1, d_2, d_3 of S such that the (Euclidean) distances between each pair of d_{1z}, d_{2z}, d_{3z}, are large enough, where d_{iz}

is the projection of d_i onto the (z=0)-plane. Since S is a digital plane, there is a convex subset T of S that contains d_1, d_2, d_3 and is a convex digital plane element. Let p be the plane given by ax+by+cz+e=0 such that the distance between T and p is the minimum. Assume without loss of generality that $0 \leq a \leq b \leq c$. We have the following two cases:

Case 1 The distance between T and p is 0.

(i) 0 = a = b

Every point of T lies on the (z=k)-plane, where k= -e/c. Consider point d = (i,j,k) which is an interior point of T. Since d has four 4-neighbors in $T \cap D_{z=k} \subset S \cap D_{z=k}$, both (i,j,k+1) and (i,j,k-1) are not points of S. Hence, there is no other point of S on the grid line which is parallel to the Z-axis and passes d because S is a digital plane. Therefore, the projection of the points of S whose projection onto the (z=0)-plane is an interior point of S_z is one-to-one. If S_z has no boundary points, then S_z is the set of all digital points of the (z=0)-plane. Thus, the projection of S onto the (z=0)-plane is one-to-one and onto the set of its digital points. Suppose S_z has a boundary point d=(i,j,k). If neither (i,j,k+1) nor (i,j,k-1) is a point of S, then d is a boundary point of S, which is a contradiction. Assume that (i,j,k+1) is a point of S. Since S is a digital plane (i,j,h+2) is not a point of S. Assume without loss of generality that (i,j+1,k) is a point of S. Then each of (i,j,k), (i,j+1,k) and (i,j,k+1) has two

8-neighbors that are mutually 8-neighbors in $S \cap D_{x=i}$. Thus, none of $(i+1,j,k)$, $(i-1,j,k)$, $(i+1,j+1,k)$, $(i-1,j+1,k)$, $(i+1,j,k+1)$ and $(i-1,j,k-1)$ are points of S. Thus, $(i,j,k+1)$ is a boundary point of S, which is a contradiction. So $(i,j,k+1)$ is not a point of S. Similarly, $(i,j,k-1)$ is not a point of S. This again is a contradiction. Thus, S has no boundary point, and the projection of S onto the $(z=0)$-plane is one-to-one and onto the set of digital points of the $(z=0)$-plane.

(ii) $0=a$ and $b=c$, or $a=b=c$.

Every point of T lies on plane $y+z=k$ or $x+y+z=k$, where $k= -e/c$. Arguments that are similar to but a little more involved than that for (i) prove that the projection of S onto the $(z=0)$-plane is one-to-one and onto the set of the digital points of the $(z=0)$-plane.

Case 2 The distance between T and p is not zero.

Translate p downward parallel to the Z-axis by the minimum distance so that every point of T lies above p. Let p' be the plane obtained by translating p upward parallel to the Z-axis by a distance of 1. Then the points of T lie between p and p', some on p but none on p'. Consider point $d=(i,j,k)$ which is an interior point of T. Since $a<c$, d has at least three 8-neighbors in $T \cap D_{z=k} \subset S \cap D_{z=k}$. Thus, both $(i,j,k+1)$ and $(i,j,k-1)$ are not points of S and no point of S other than d is on the grid line that passes d and is parallel to Z-axis. Therefore, the projection of the points of S whose projections

onto the (z=0)-plane are interior points of S_z is one-to-one. Thus, every point of S lies between p and p', possibly on p but not on p', because S is a digital plane. Suppose that S_z has a boundary point d=(i,j,k). Since (i,j,k-1) lies below p and (i,j,k+1) lies on or above p', they are not points of S. Thus, d is a boundary point of S, which is a contradiction. Therefore, the (z=0)-plane is a coordinate plane such that the projection of S is one-to-one and onto the set of its digital points. □

Lemma 10

Digital planes have the chordal triangle property.

Proof

Let S be a digital plane and d_1, d_2, d_3 any three points of S. By the above lemma, there is a coordinate plane, say the (z=0)-plane, such that the projection of S onto it is the set of all of its digital points. Let d_{1z}, d_{2z} and d_{3z} be the projections of d_1, d_2 and d_3 onto the (z=0)-plane. We denote by t_z the triangle whose vertices are d_{1z}, d_{2z} and d_{3z} and by T_z the set of digital points of t_z. Choose three digital points d'_{1z}, d'_{2z}, d'_{3z} on the (z=0)-plane so that the following are satisfied: The triangle t'_z whose vertices are d'_{1z}, d'_{2z}, d'_{3z} contains t_z, and no point of T_z is a boundary point of T'_z, the set of digital points of t'_z.

Let d'_1, d'_2 and d'_3 be the points of S whose projections onto the (z=0)-plane are d'_{1z}, d'_{2z} and d'_{3z}, respectively.

Since S is a digital plane, there is a finite subset R of S that contains d'_1, d'_2, d'_3 and is a convex digital plane element. Let p be a plane such that R is the digital image of a subset g of p, that is, R=I(g). Suppose that p is given by the equation ax + by + cz + e =0 and assume without loss of generality that $0 \le a \le b \le c$. Then the projection of R onto the (z=0)-plane is one-to-one. Let t' be the subset of p such that its projection onto the (z=0)-plane is t'_z and T and T' the subsets of S whose projections onto the (z=0)-plane are T_z and T'_z, respectively. Obviously, T' is a subset of R. Let t'_u and t'_b be the triangles obtained by translation of t' parallel to the Z-axis by 1/2 and -1/2, respectively. Then every point of T' is between t'_u and t'_b, possibly on t'_b but not on t'_u.

If t is the chordal triangle of d_1, d_2 and d_3, then it lies between t'_b and t'_u, possibly touching t'_b but not t'_u. Consider any point w on t. If it is a digital point, then it must be a point of T' and thus it is near S. It is easy to see that if it is not a digital point, then there is at least one point of T' which is near w. Thus, every point of t is near S. Therefore, S has the chordal triangle property. □

Lemma 11

If a simple digital surface has the chordal triangle property, then there is a coordinate plane such that the projection of the digital surface on the coordinate plane is one-to-one and onto the set of all digital points of the coordinate plane.

Proof

If a set S of digital points is a digital surface, then there is a coordinate plane, say the (z=0)-plane, such that S_z, the projection of S onto it, is not bounded by any two parallel lines as shown in the proof of Lemma 9. Let d_1, d_2, d_3 be points of S that satisfy the following: (i) The (Enclidean) distance between each pair of d_{1z}, d_{2z}, d_{3z} is as large as we wish, where d_{iz} is the projection of d_i onto the (z=0)-plane. (ii) If R is the set of all points of S whose projections onto the (z=0)-plane are points of the triangle with vertices d_{1z}, d_{2z} and d_{3z}, then R lies above the triangle t whose vertices are d_1, d_2 and d_3. Such d_1, d_2 and d_3 exist because S is a digital surface and has the chordal triangle property. Suppose that t is a subset of the plane p represented by $ax+by+cz+e=0$ and assume without loss of generality that $0 \le a \le b \le c$. First consider the case where not all the points of intersection of t with grid lines parallel to the Z-axis are digital points. Using similar arguments as in the proof of Lemma 9, there is at most one point of R on each grid line parallel to Z-axis. If p' is the plane obtained from p by translating it in parallel to the Z-axis by distance 1, then all the points of S must lie between p and p', possibly on p but not on p', because S has the chordal triangle property. Again by similar arguments, S_z has no boundary points. So the projections of S onto the (z=0)-plane is one-to-one and onto the set of all digital points of the (z=0)-plane. The case where all the points of intersection of t with grid lines parallel to the Z-axis are digital points, that is, $0 = a = b$, $0 = a$ and $b = c$, or $a=b=c$, may be treated similarly. □

Lemma 12

If a simple digital surface has the chordal triangle property, then it is a digital plane.

Proof

Let S be a simple digital surface and suppose that it has the chordal triangle property. By the above lemma, there is a coordinate plane such that the projection of S onto it is one-to-one and onto the set of all of its digital points. Let d_1, d_2 and d_3 be any three points of S and d'_1, d'_2 and d'_3 be the projections of d_1, d_2 and d_3 onto the coordinate plane, respectively. We denote by t' the triangle whose vertices are d'_1, d'_2 and d'_3 and by T' the set of digital points of t'. Let T be the subset of S whose projection onto the coordinate plane is T'. Let p be a support of T such that the distance between p and T is not greater than the distance between any support of T and T. Suppose that p is given by equation $ax+by+cz+e=0$ and assume without loss of generality that $0 \le a \le b \le c$. Assume that the distance between p and T is less than 1. Then the vertical distance from every point of T to p is less than 1. Assume without loss of generality that T lies above p. Let q be the plane obtained by translating p upward parallel to the Z-axis by a distance of 1/2. Then the vertical distance from any point of T above q is less than 1/2 and from any point of T below q is less than or equal to 1/2. Thus, by Lemma 1 T is the digital image of a subset of q. Hence, S is a digital plane by definition. Now assume that the distance between p and T is greater than or equal to 1. Suppose that all the points of T on p are collinear. (There must be at least one point of T on p since otherwise p may be translated parallel to the Z-axis toward T to obtain a plane with shorter distance between T and the plane.) Then there are three points, d_4, d_5 and d_6, of T such that the distances between T and those points are equal to the distance between p and T. Moreover, one of the points of T on p, say d, is such that d', the projection of d onto the (z=0)-plane, is an interior point of the triangle whose vertices are d'_4, d'_5 and d'_6 which are the projections of d_4, d_5 and d_6, respectively. For, otherwise by slight rotation of p, we may obtain a support T such that the distance between the new support and T is smaller than that between p and T, which is a contradiction. Consider the chordal triangle of T whose vertices are d_4, d_5 and d_6. Let w be the point of the chordal triangle whose x- and y-coordinates are the same as those of d, that is, w and d are on the same coordinate line parallel to the Z-axis. Since the distance between w and d is at least 1 and d is the only point of T on the coordinate line, w is not near T. Hence, S does not have the chordal triangle property, which is a contradiction.

Suppose now that there are three noncollinear points, day d_4, d_5 and d_6, of T on p. Assume that there is a point, say d, of T such that the distance between d and p is the same as the distance between T and p, and d' is an interior point of the triangle with vertices d_7, d_8 and d_9 which is on the same coordinate line parallel to the Z-axis as d is not near T. Thus, S does not have the chordal triangle property, which is a contradiction. So assume that for every point of T whose distance to p is equal to the distance between T and p, its projection onto the coordinate plane is not an interior point of the triangle with vertices d'_4, d'_5 and d'_6. Then there are at least three noncollinear points, say d_7, d_8 and d_9, of T such that the distances from them to p are equal to the distance between p and T. If not, by slightly rotating p, we may obtain a support such that the distance between the new support and T is smaller than that between p and T. If any of d'_4, d'_5 and d'_6 is an interior point of the triangle whose vertices are d'_7, d'_8 and d'_9, then S does not have the chord property. Thus, none of d'_4, d'_5 and d'_6 is an interior point of the triangle. Let d'_{10}, d'_{11} and d'_{12} be three points on the coordinate plane such that $d'_{10}d'_{11}$, $d'_{11}d'_{12}$ and $d'_{12}d'_{10}$ are parallel to $d'_4d'_5$, $d'_5d'_6$ and $d'_6d'_4$, respectively, and d'_4, d'_5 and d'_6 are on $d'_{11}d'_{12}$, $d'_{12}d'_{10}$ and $d'_{10}d'_{11}$, respectively (see Figure 5). Then it is easy to see that d'_7, d'_8 and d'_9 are such points, where $d'_7d'_8$, $d'_8d'_9$ and $d'_9d'_7$ are parallel to $d'_4d'_5$, $d'_5d'_6$ and $d'_6d'_4$, respectively. Let d_{10}, d_{11} and d_{12} be the points of T whose projections are d'_{10}, d'_{11} and d'_{12}, respectively. If all of them are points of p, then the chordal triangle of T with these three points as its vertices is not near S because point w of the chordal triangle whose projection onto the coordinate plane is d'_7 is not near T.

Similarly, if none of the three points are on p then the chordal triangle is not near S. Suppose that two of the three, say d_{10} and d_{11}, are on p. Then the chordal triangle of T whose vertices are d_4, d_{10} and d_{11} is not near S. Next, suppose that two of the three, say d_{10} and d_{11}, are not on p. Then the chordal triangle of T whose vertices are d_{10}, d_{11} and d_9 is not near S.

Therefore, if the distance between p and T is greater than or equal to 1, S does not have the chordal triangle property, which is a contradiction to the hypothesis of the lemma. Hence, the distance between p and T is less than 1 and so S is a digital plane. □

Theorem 13

A simple digital surface is a digital plane if and only if it has the chordal triangle property.

Because of Lemma C, we have the following theorem, which is a corollary to the above theorem.

Theorem 14

A simple digital surface is a digital plane if and only if it is digitally convex.

6. Algorithms

In this section algorithms are presented that determine whether or not a digital surface element is a digital plane element and a convex digital plane element.

The definitions of digital plane element and convex digital plane element do not easily lead to the design of such algorithms. Thus, we need some characterizations of the above digital objects which lend themselves to development of these algorithms. In the following we obtain such characterizations.

Theorem 15

A finite digital surface is a digital plane element if and only if there is a support such that the distance between the finite digital surface and the support is less than 1.

Proof

Suppose that S is a finite digital surface and assume that plane p is a support of S such that the distance between the two is less than 1. Let p be given by $ax+by+cz+e=0$ and assume $0\le a\le b\le c$. Then the vertical distance from any point of S to p is less than 1. Let p' be the plane that is obtained from p by upward translation parallel to the Z-axis by a distance of 1/2. Then S is a subset of I(p'), the digital image of p'. Thus, we may find a connected subset g of p' whose digital image is S. Therefore, S is a digital plane element.

Now suppose that S is a digital plane element. Then there is a plane p such that g is a connected finite subset of p whose digital image is S. Let p be given by $ax+by+cz+e=0$ and assume

$0 \leq a \leq b \leq c$. Then the vertical distance to p from every point of S that lies above p is less than 1/2 and that from every point of S that lies below p is at most 1/2. If p' is the plane that is obtained from p by translation parallel to the Z-axis downward by a distance of 1/2. then p is a support of S and the distance between S and p is less than 1. □

Let S be a finite set of digital points and H(S) its convex hull. If p is a plane such that a face F of H(S) is a subset of p, then p is called the plane of the face F.

Theorem 16

A finite digital surface is a digital plane element if and only if there is a face of the convex hull of the digital surface such that the distance between the digital surface and the plane of the face is less than 1.

Proof

Let S be a finite digital surface and H(S) its convex hull. Suppose there is a face F of H(S) such that the distance between S and p, the plane of the face F, is less than 1. Obviously, p is a support of S. Hence, by Theorem 15 S is a digital plane element.

Now suppose that S is a digital plane element. By Theorem 15, there is a support of S such that the distance between S and the support is less than 1. Let p be a support of S such that the distance between p and S is the minimum. If p contains three points of S that are not collinear, then it is the plane of a face of H(S). Otherwise, there is a support p' of S such that p and p' are parallel, S lies between the two and there are three points of S on p' that are not collinear.

For, if there is no such p', by rotation of p we may obtain another support p" of S such that the distance between p" and S is smaller than that between p and S, which is a contradiction. Hence, p' is the plane of a face of H(S). □

<u>Corollary 17</u>

A finite digital surface is a convex digital plane element if and only if its projections onto the coordinate planes are convex digital regions and there is a face of its convex hull such that the distance between the digital surface and the plane of the face is less than 1.

We now present an algorithm to determine whether or not a finite set of digital points is a convex digital plane element.

<u>Algorithm</u> CONVEX-PLANE(S)

Given a finite set S of digital points, algorithm CONVEX-PLANE returns 'True' if S is a convex digital plane element and 'False' otherwise.

<u>Step 1</u> Check if there is a coordinate plane onto which the the projection of S is one-to-one.
<u>If</u> not <u>then</u> print (<u>False</u>) and return.

<u>Step 2</u> <u>For</u> each coordinate plane <u>do</u>
project S onto the coordinate plane;
determine if the projection is a convex digital region;
<u>if</u> not <u>then</u> print (<u>False</u>) and return.

<u>Step 3</u> Construct H(S), the convex hull of S.

Step 4 Find a face of H(S) such that the distance between S and F, the plane of the face, is less than 1.
If found then print (True) else print (False).
Return.

Algorithm PLANE(S) that determines whether or not a finite set S of digital points is a digital plane element may be obtained from algorithm CONVEX-PLANE by simply removing Step 2.

By Lemma 1, Corollary 4, Theorem 16 and Corollary 17, it is immediate that these algorithms work correctly. To analyze their computational complexities, detailed descriptions of each step and the data structures used in the algorithms are required. For simplicity we assume that a finite set S of digital points is a subset of the set of N^3 digital points in the cube with edges of length n. S is represented by a run length code such that RC(i,j), $1 \le i \le n$ and $1 \le j \le n$, is a finite sequence of run lengths of 0's and 1's. Thus, RC(i,j) = $(\ell_{ij0}, \ell_{ij1}, \ldots, \ell_{ijr})$ means that the (i,j)th row of the cube is composed of a run of 0's (points not in S) of length ℓ_{ij0} followed by a run of 1's (points in S) of length ℓ_{ij1} and so on.

(1) Is there a coordinate plane onto which the projection of S is one-to-one?

If for each i and j, RC(i,j) has at most one run of 1's of length 1, then the projection of S onto the Z-coordinate plane is one-to-one. This takes $O(n^2)$ time. To see if the projection of S onto the X-coordinate plane is one-to-one, we do the following for each j, $1 \le j \le n$: Let A[1:n] be a linear array of size n and initialized to 0. For i from 1 to n, set $A(k) = 1$ if (i,j,k) is a point of S, where $1 \le k \le n$. Use RC(i,j) to check if (i,j,k) is a point of S. If A(k) is already 1 when we try to set A(k) to 1, then the projection of S onto the X-coordinate plane is not one-to-one. This also takes $O(n^2)$ time.

Whether or not the projection of S onto the Y-coordinate plane is one-to-one may be checked similarly. Thus, Step 1 of the algorithm takes $O(n^2)$ time and O(n) work space.

(2) Are the projections of S onto the coordinate planes convex digital regions?

Consider the projection of S onto the X-coordinate plane. For each j, $1 \le j \le n$, create R(j), the run length code of 0's and 1's on the row y=j of the projection of S, by using the RC(i,j)'s for all i. Each R(j) should have at most one run of 1's, since otherwise the projection is not a convex digital region. This procedure takes $O(n^2)$ time and O(n) work space. Next use algorithm CONVEX in [6] to determine whether or not the projection of S onto the X-coordinate plane is a convex digital region. This algorithm is based on Lemma A and takes O(n) time.

The projections of S onto the other coordinate planes may be checked for digital convexity similarly. Hence, Step 2 of algorithm CONVEX-PLANE takes $O(n^2)$ time and O(n) space.

(3) Construction of the convex hull of S.

Use the convex hull algorithm in [13] to construct H(S). Since S has at most n^2 points, the computing time required is $O(n^2 \log n)$ and the work space required is $O(n^2)$.

(4) Is there a face of H(S) such that the distance between S and the plane of the face is less than 1?

For each face of H(S) constructed in Step 3, obtain the plane of the face. Then find the distance between S and the plane using the definition given in Section 2. As soon as we find a face of H(S) that satisfies the condition, Step 4 is completed. There are at most n^2 faces of H(S) and for each face, to find the distance between S and the plane of the face takes $O(n^2)$ time. Thus, Step 4 requires $O(n^4)$ time.

Summarizing the arguments given above we obtain:

Theorem 18

Algorithm PLANE(CONVEX-PLANE) determines whether or not a finite set of digital points is a (convex) digital plane element and has time complexity of $O(n^4)$ and space complexity of $O(n^2)$.

7. Conclusion

This paper is a result of a continuing effort to develop a theory of digital geometry in 2- and 3-dimensional space. Surfaces being an important object of study in geometry, digital surfaces, and in particular, the planarity of digital surfaces, is the subject of this work.

Digital surfaces were defined; the definition was derived from intuition, and partially justified by the fact that the digital image of a plane is always a digital surface. Further justification is based on the result that digital convexity is a necessary and sufficient condition for a digital surface to be a digital plane. This is an important property of Euclidean planes.

However, there are still many properties that we might have wanted to be necessary and sufficient conditions for a digital surface element to be a digital plane element, but are not. Two such properties are digital convexity and the line property. It is an open question whether there are other definitions of digital surface and digital plane for which digital planes and digital plane elements would enjoy more geometric properties that are enjoyed by Euclidean counterparts.

Finally we were able to characterize (convex) digital plane elements by a simple geometric property. This led to the development of relatively simple algorithms to determine whether or not a finite set of digital points is a (convex) digital plane element. Even though the algorithms are conceptually simple, they have costly time and space complexities. Development of algorithms with more favorable time and space complexities is desirable.

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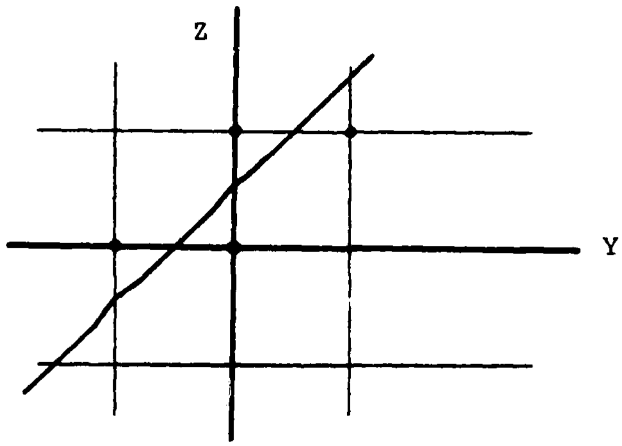

Fig. 1. A surface with curvature near 0 everywhere whose digital image is not a digital surface.

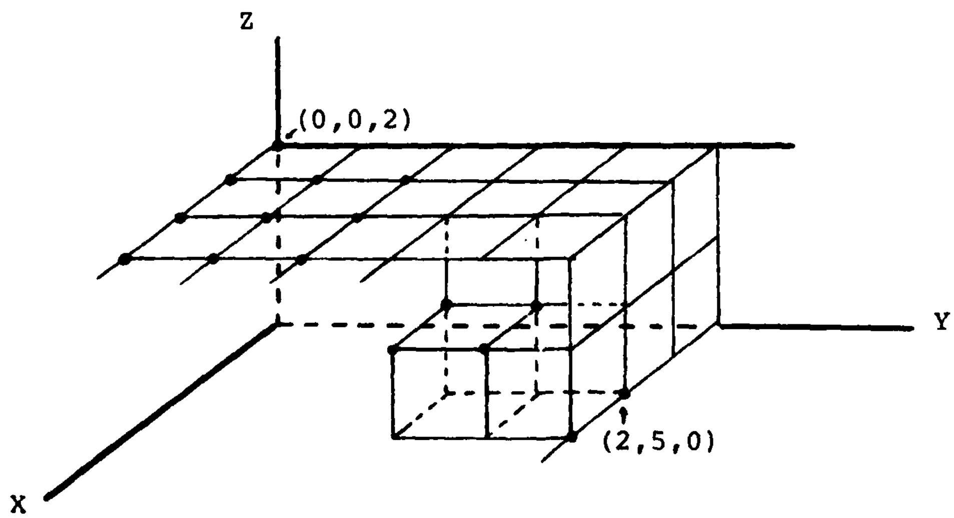

Fig. 2. A convex digital plane element R which does not have the chord property and hence does not have the chordal triangle property.

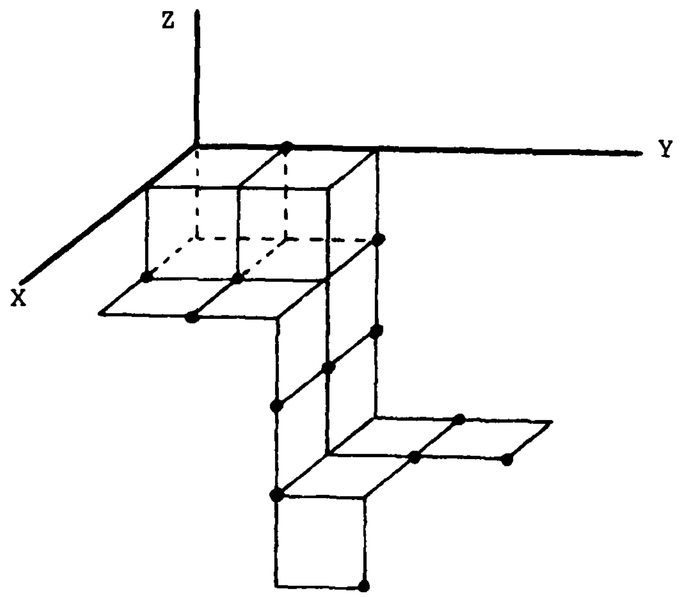

Fig. 3. A digital surface element R′ which has the chordal triangle property but is not a plane element.

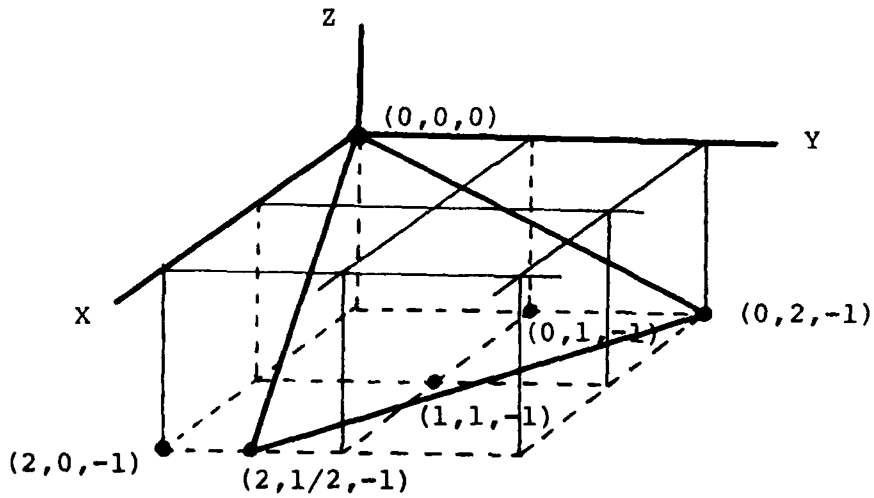

Fig. 4. A convex plane element whose digital image is not a convex digital plane element.

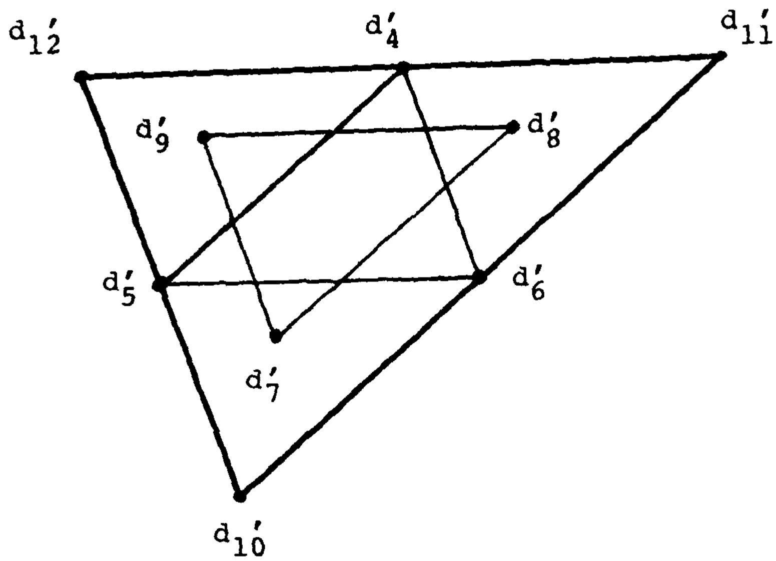

Figure 5

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Definitions of 3D digital surface and plane are introduced. Many geometric properties of these objects are examined. In particular, it is shown that digital convexity is neither a necessary nor a sufficient condition for a digital surface element to be a convex digital plane element, but it is both necessary and sufficient for a digital surface to be a digital plane. Also algorithms are presented to determine whether or not a finite set of digital points is a (convex) digital plane element.

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